K-theoretic Quasimap Wallcrossing

Yang Zhou,
SCMS, Fudan
j.w. Ming Zhang

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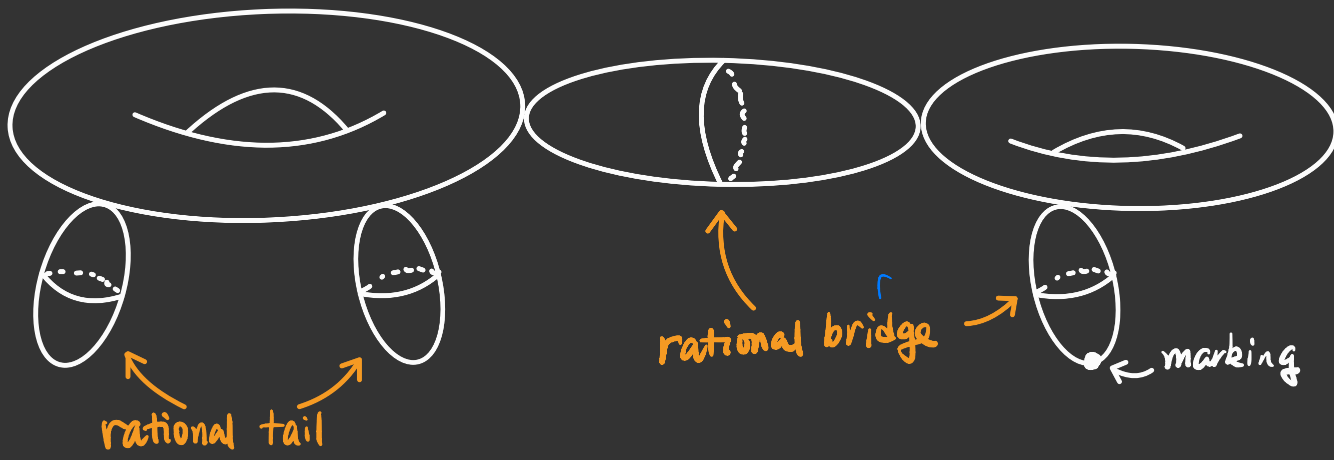


<u>Defn</u>: (Ciocan-Fontanine–Kim–Maulik, '14)

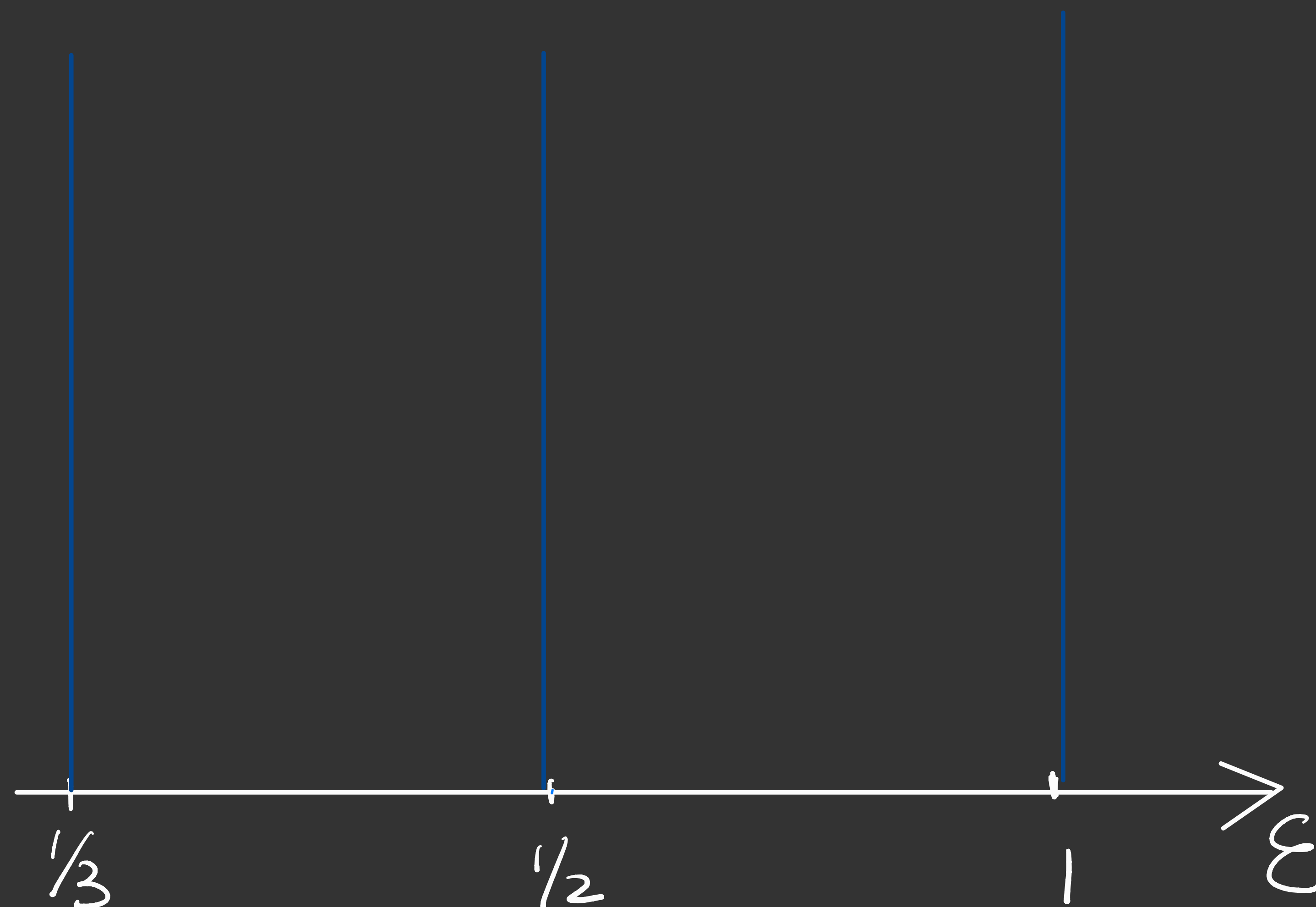
A quasi-map is ε-stable ($\varepsilon \in \mathbb{Q}_{>0}$) if

- Every rational bridge has degree > 0
- Every rational tail has degree $> \frac{1}{\varepsilon}$
- Every base point has length $\leq \frac{1}{\varepsilon}$

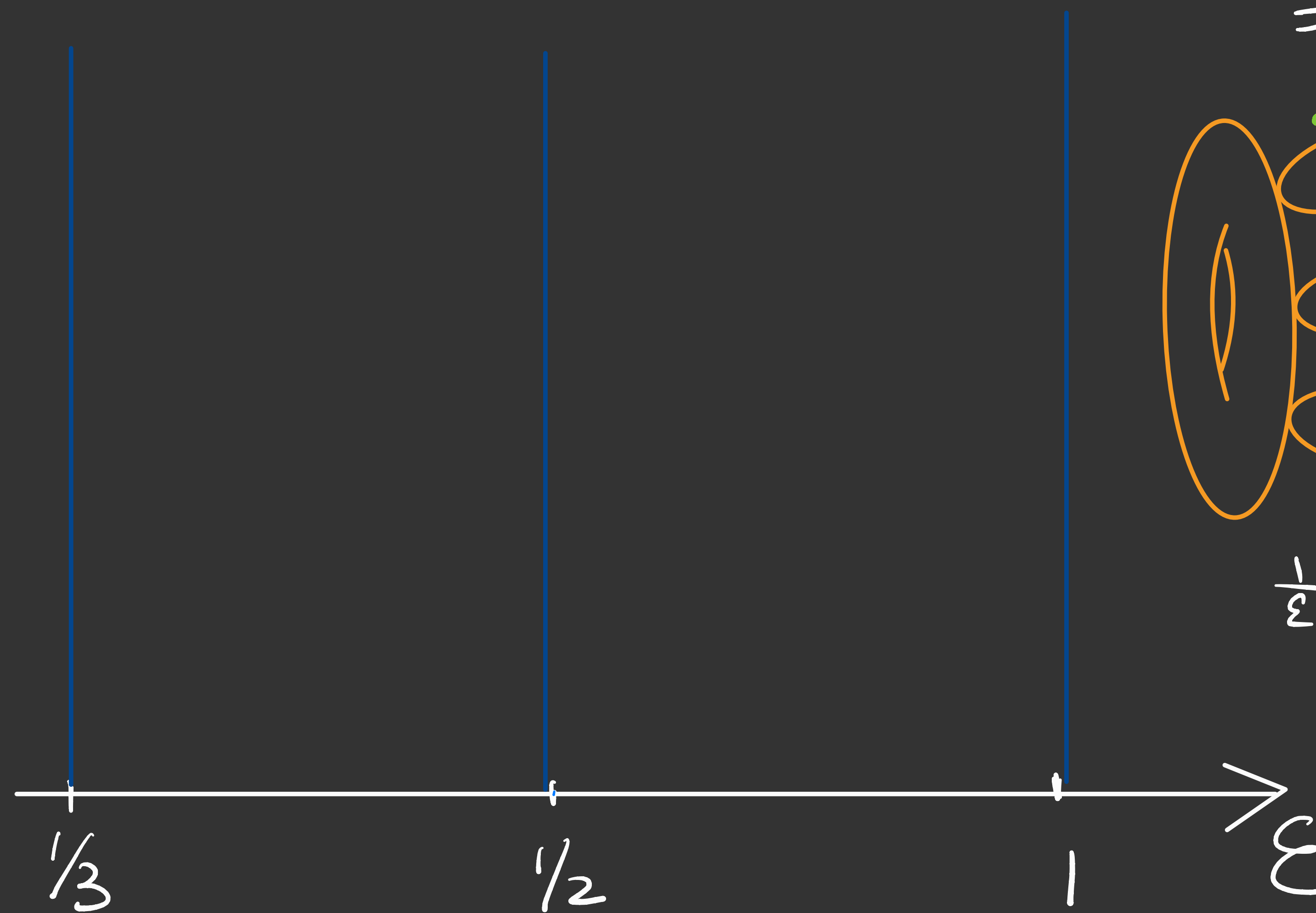
$+$

$\deg > \frac{2}{\varepsilon}$.

Wall-and-chamber structure:

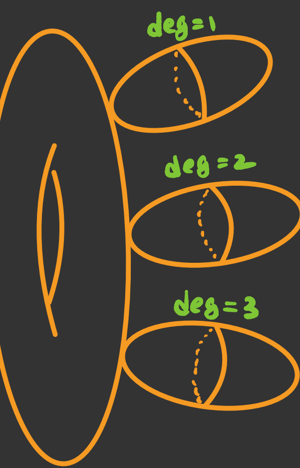


Wall-and-chamber structure:



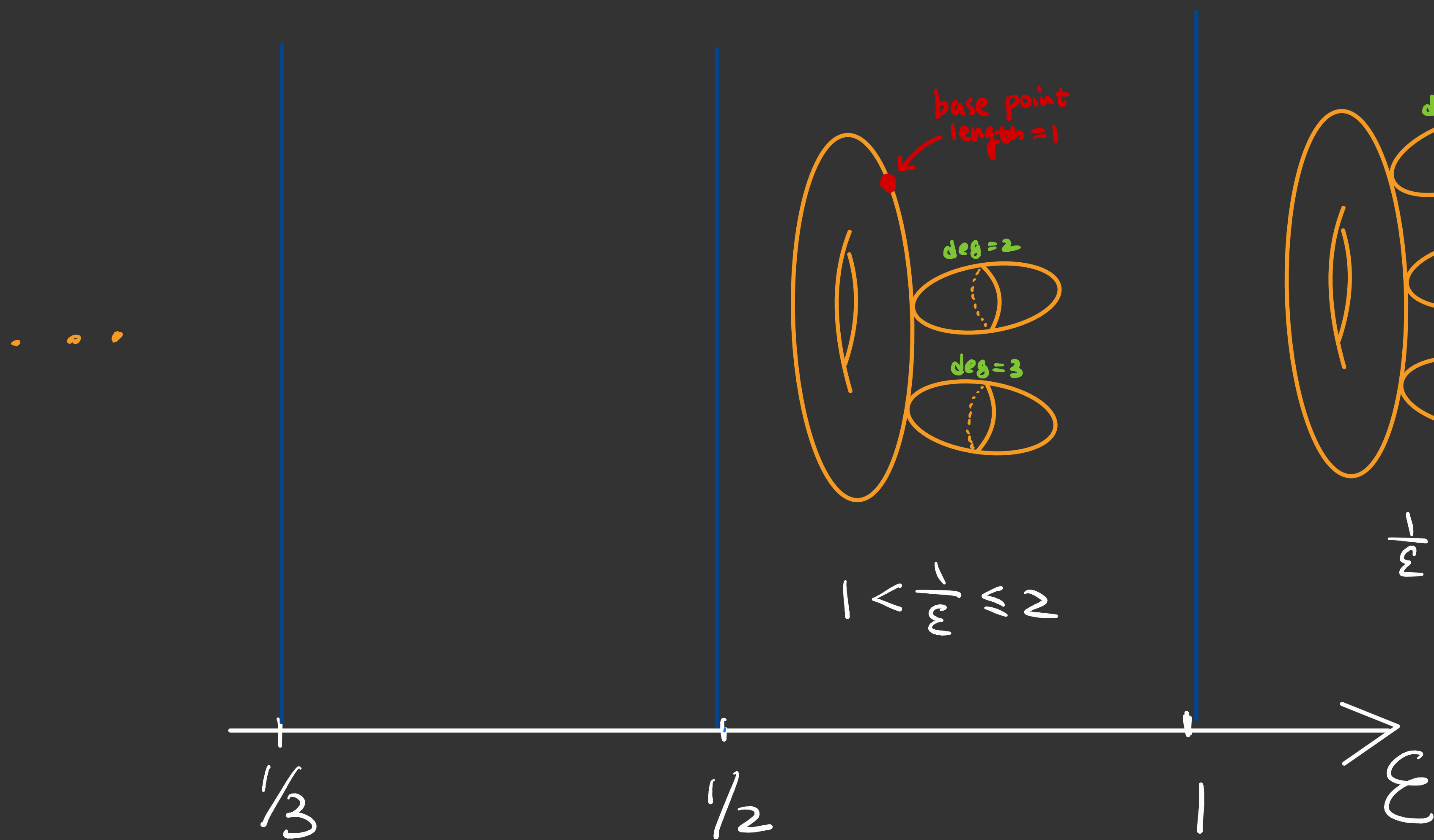
$\Longrightarrow$ stable map

$\Rightarrow$ GW theory

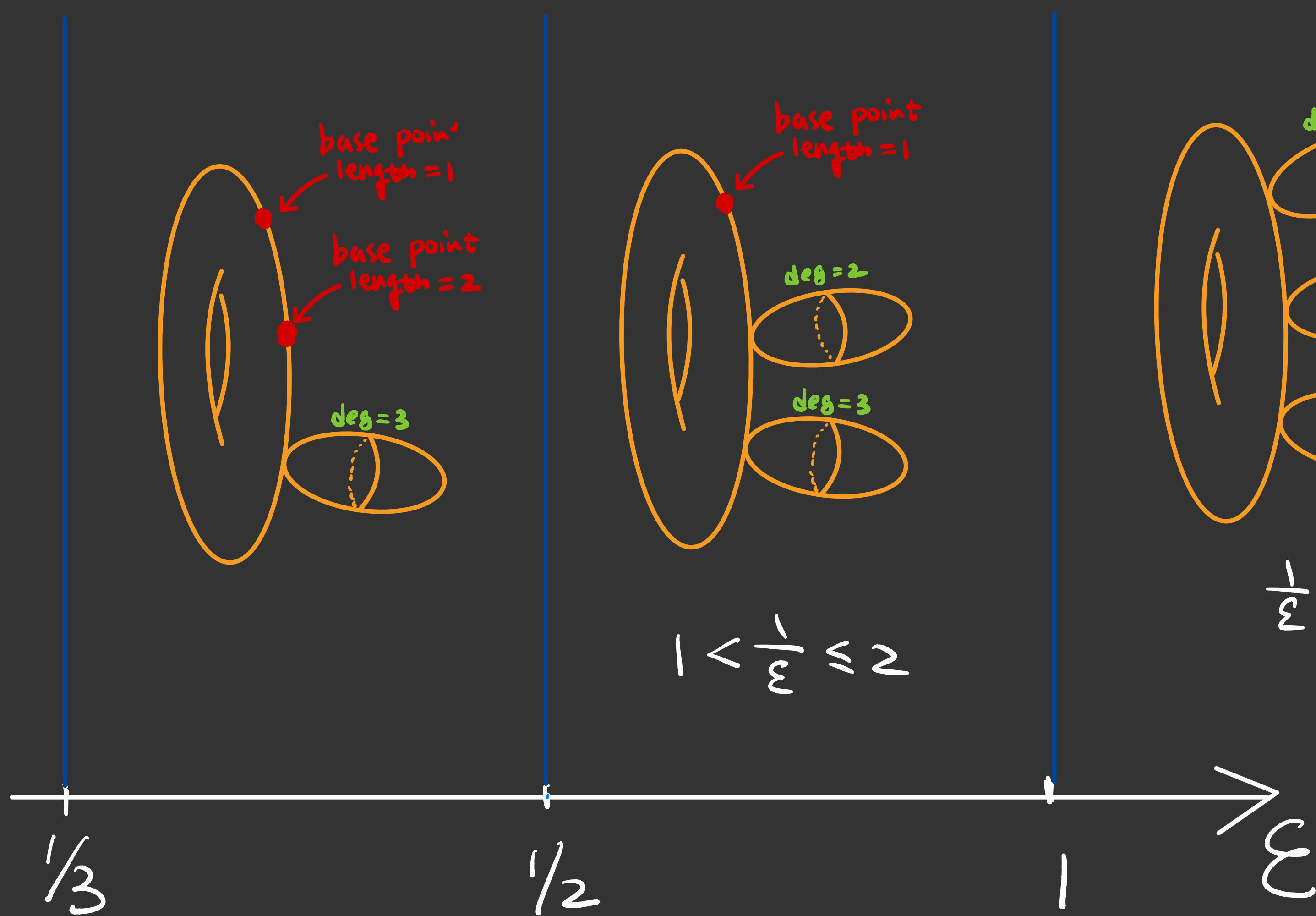


$\frac{1}{\varepsilon} < 1$

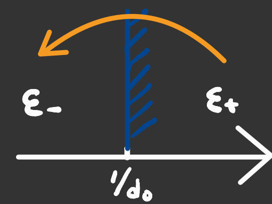
Wall-and-chamber structure:



Wall-and-chamber structure:



Cohomological Wall-crossing formula.



$$\begin{tikzcd}
Q^{\varepsilon_+}(X) \arrow[r, dashed] \arrow[d] & Q^{\varepsilon_-}(X) \arrow[d] \\
Q^{\varepsilon_+}(\mathbb{P}^N) \arrow[r] & \boxed{Q^{\varepsilon_-}(\mathbb{P}^N)}
\end{tikzcd}$$

Conjecture: (Ciocan-Fontanine — Kim)

$$[Q^{\varepsilon_-}_{g,n}(X,\beta)]^{vir} - [Q^{\varepsilon_+}_{g,n}(X,\beta)]^{vir}$$

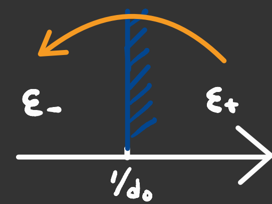
$$\text{"="} \sum_{k\geq 1} \sum_{\vec{\beta}} \frac{1}{k!} \prod_{i=1}^{k} ev_{n+i}^{*}\, \mu_{\beta_i}(z)\Big|_{z=-\psi_{n+i}} \cap [Q^{\varepsilon_+}_{g,n+k}(X,\beta')]^{vir}$$

where $\vec{\beta} = (\beta', \beta_1, \cdots, \beta_k)$, $\beta = \beta' + \beta_1 + \cdots + \beta_k$, $\deg(\beta_i) = d_0$

$$\mu_{\beta_i} \longleftarrow \left\{ \mathbb{P}^1 \overset{\beta_i}{\dashrightarrow} X \right\}$$

base point
length $= d_0$

K-theoretic Wall-crossing formula.



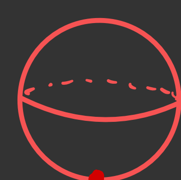
Thm (M. Zhang – Z)

$$\mathcal{O}^{vir}_{Q^{\varepsilon_-}_{g,n}(X,\beta)} - \mathcal{O}^{vir}_{Q^{\varepsilon_+}_{g,n}(X,\beta)}$$

$$\text{“}=\text{”} \sum_{k\geq 1} \sum_{\vec{\beta}} \prod_{i=1}^{k} ev_{n+i}^{*}\, \mu_{\beta_i}(L) \cap \mathcal{O}^{vir}_{[Q^{\varepsilon_+}_{g,n+k}(X,\beta')/S_k]}$$

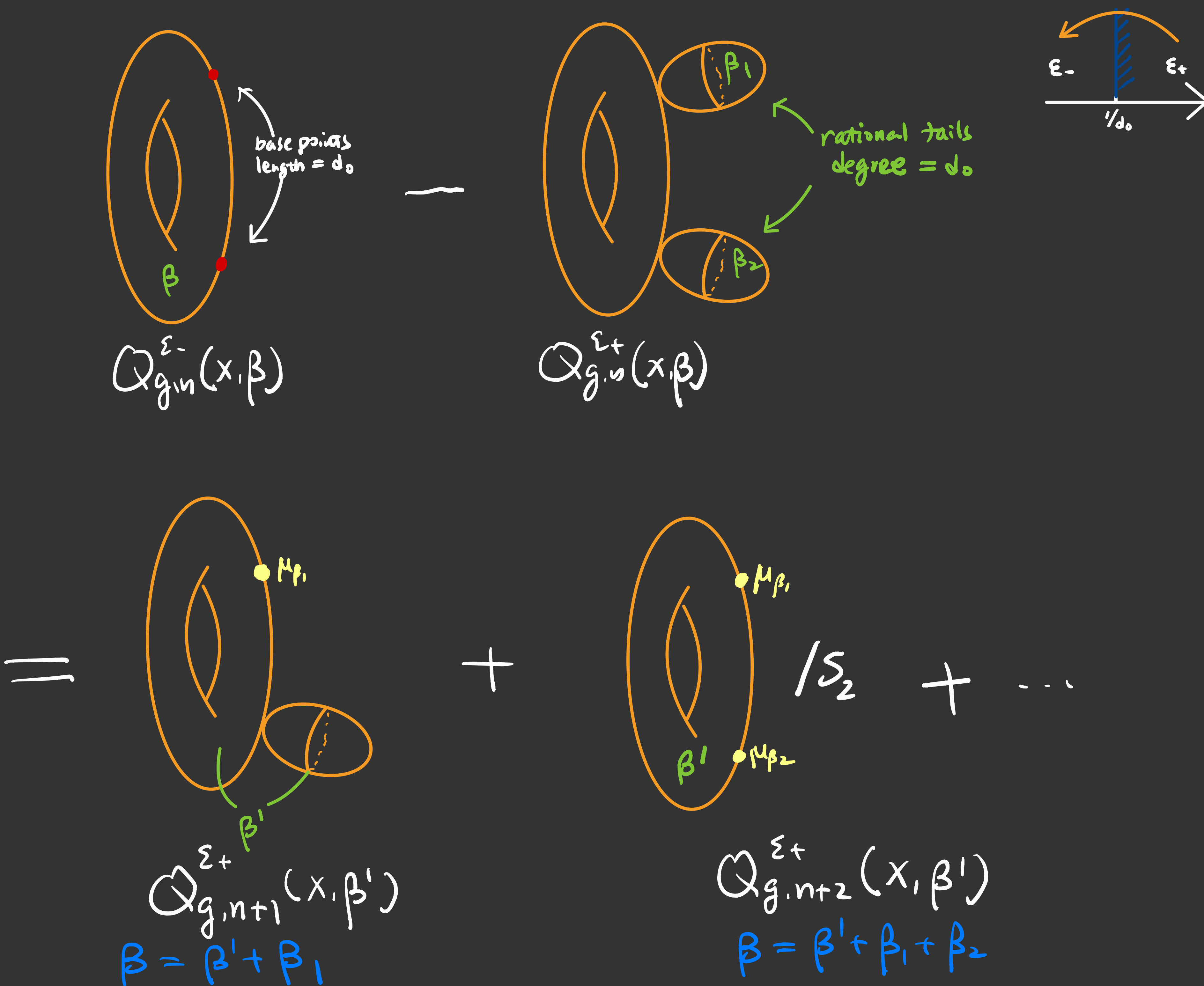
Basic W-C

where $\vec{\beta} = (\beta', \beta_1, \cdots, \beta_k)$, $\beta = \beta' + \beta_1 + \cdots + \beta_k$, $\deg(\beta_i) = d_0$

$\mu_{\beta_i} \longleftarrow \{ \mathbb{P}^1$  $\xrightarrow{\beta_i} X \}$

base point
length $= d_0$

+ small result in the case: $\varepsilon = (1/d)^+$, $g=0$, $n=1$,  irred.



base points
length = d_0
β
$Q^{\varepsilon_-}_{g,n}(X,\beta)$
$-$
β_1
β_2
rational tails
degree = d_0
$Q^{\varepsilon_+}_{g,n}(X,\beta)$
ε_-
ε_+
$1/d_0$
$=$
μ_{β_1}
β'
$Q^{\varepsilon_+}_{g,n+1}(X,\beta')$
$\beta = \beta' + \beta_1$
$+$
μ_{β_1}
μ_{β_2}
β'
$/S_2$
$+ \cdots$
$Q^{\varepsilon_+}_{g,n+2}(X,\beta')$
$\beta = \beta' + \beta_1 + \beta_2$

Packaging into generating functions:

$$F_g^{\epsilon}(\mathbf{t}(q)) := \sum_{n=0}^{\infty} \sum_{\beta \geq 0} Q^{\beta} \langle \mathbf{t}(L), \ldots, \mathbf{t}(L) \rangle_{g,n,\beta}^{S_n, \epsilon}$$

where $\mathbf{t} \in K(\bar{I}X) \otimes \Lambda\,[q]$

<u>Thm</u> (M. Zhou – Z)

$(g \geq 1)$ $F_g^{\epsilon}(\mathbf{t}(q)) = F_g^{+\infty}(\mathbf{t}(q) + \mu^{\geq \epsilon}(Q, q))$

$(g = 0)$ same modulo linear terms in $\mathbf{t}$.

Genus 0 result:

$$J^{\varepsilon}_{S_\infty}(\mathbf{t}(q), Q) :=$$

$$1 - q + \mathbf{t}(q)$$

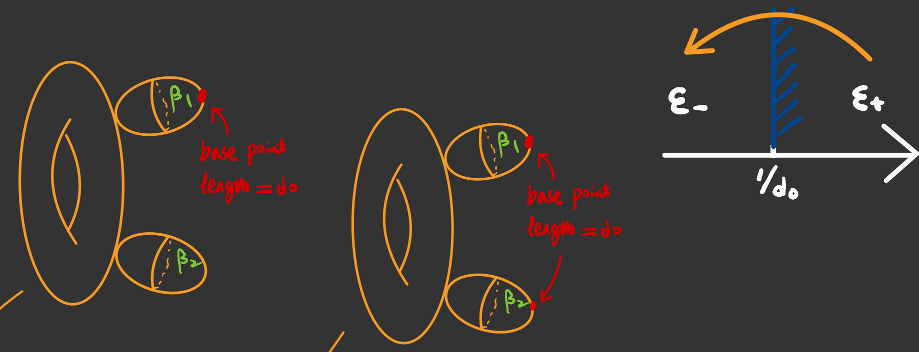
$$+ (1-q^{-1})(1-q) \cdot \sum_{0<\beta\le 1/\varepsilon} Q^{\beta} (\underline{\check{ev}_{*}})_* \left(\frac{\mathcal{O}^{vir}_{F_\beta}}{\lambda^{\mathbb{C}^*}_{-1}(N^{vir}_{F_\beta})} \right)$$

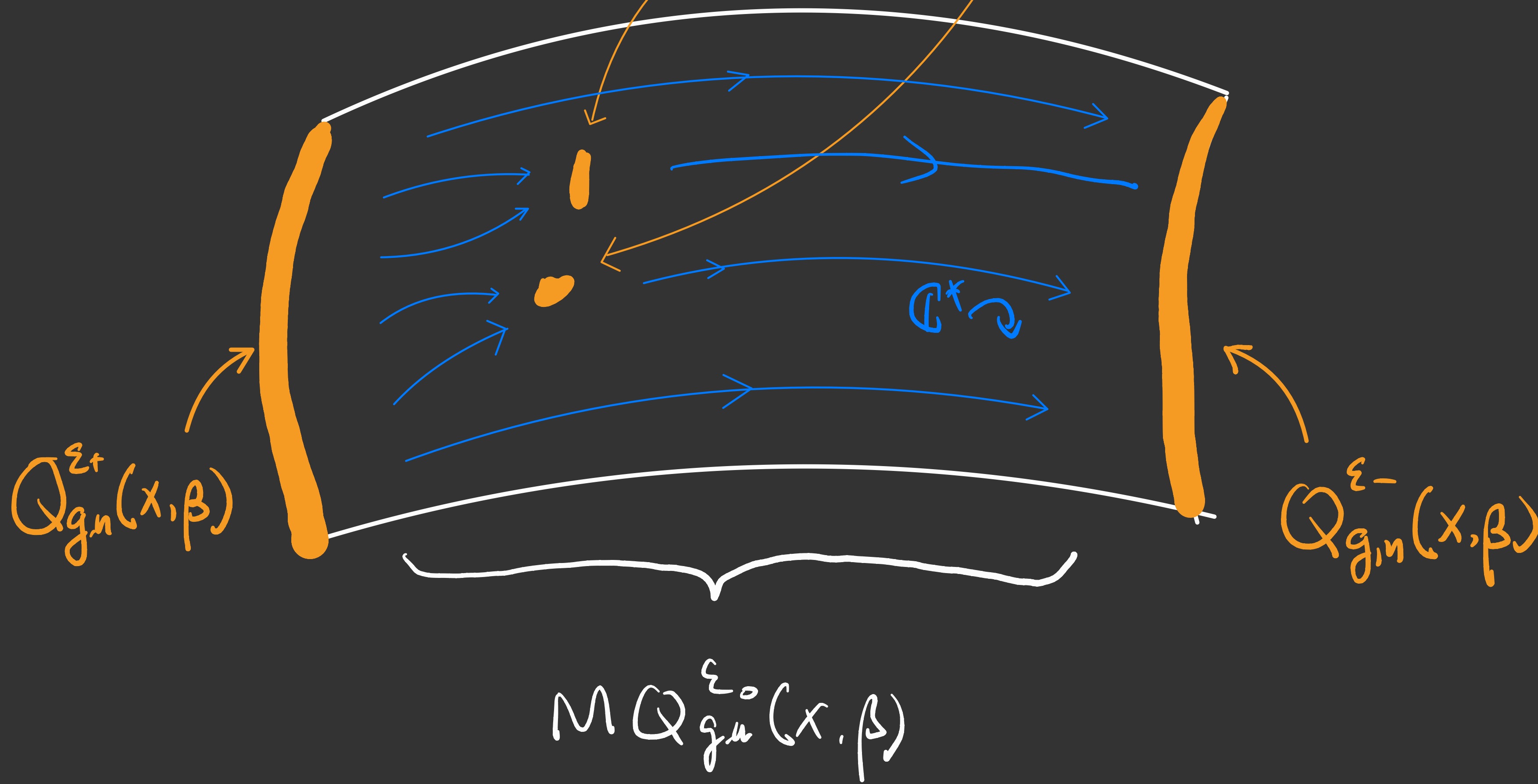
$$+ \sum_{\substack{(k\ge 1, \beta\ge 0),\\ (k,\beta)\ne(1,0) \text{ or} \\ k=0,\ \deg(\beta)>1/\varepsilon}} Q^{\beta} (\underline{\check{ev}_1})_* \left(\frac{\mathcal{O}^{vir}_{[Q^{\varepsilon}_{0,1+k}(X,\beta)/S_k]}}{1-qL_1} \cdot \prod_{i=1}^{k} ev_i^*(\mathbf{t}(L_i)) \right)$$

<u>Thm</u> (M. Zhang – Z)

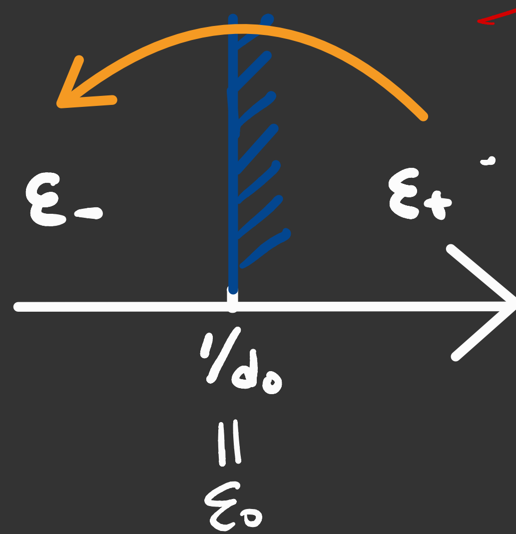
$$J^{\infty}_{S_\infty}(\mathbf{t}(q) + \mu^{\ge\varepsilon}(Q,q), Q) = J^{\varepsilon}_{S_\infty}(\mathbf{t}(q), Q)$$

Idea of proof:





- Allows base points of length d_0
- ~~Disallows rat'l tails of degree d_0~~

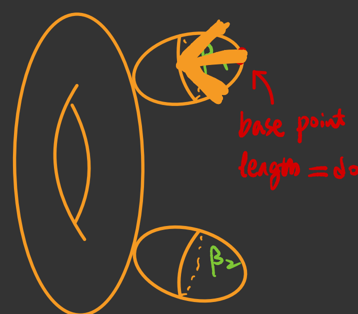


- ~~Disallows base points of length d_0~~
- Allows tails of degree d_0.

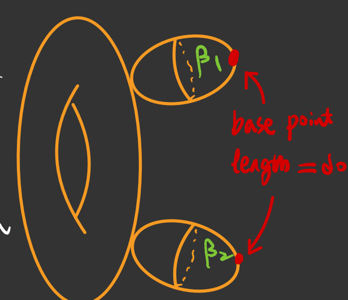
Key observation:

If we allow both degree $= d_0$ rational tails and length $= d_0$ base points

Then

Aut(

) $= \infty$,

Aut(

) $= \infty$,

⋮

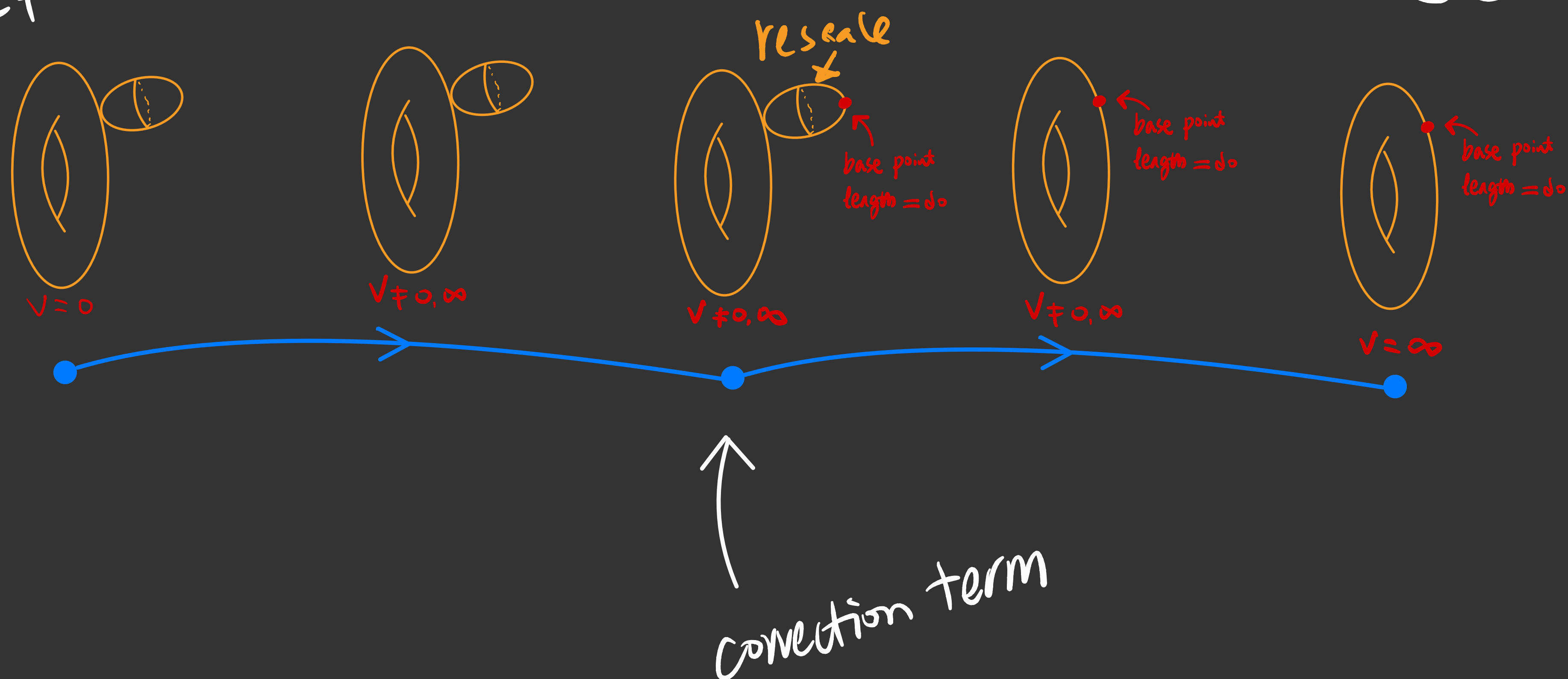
$V \in \left(\Theta_1 \otimes \Theta_2\right)^\vee \cup \{\infty\}$

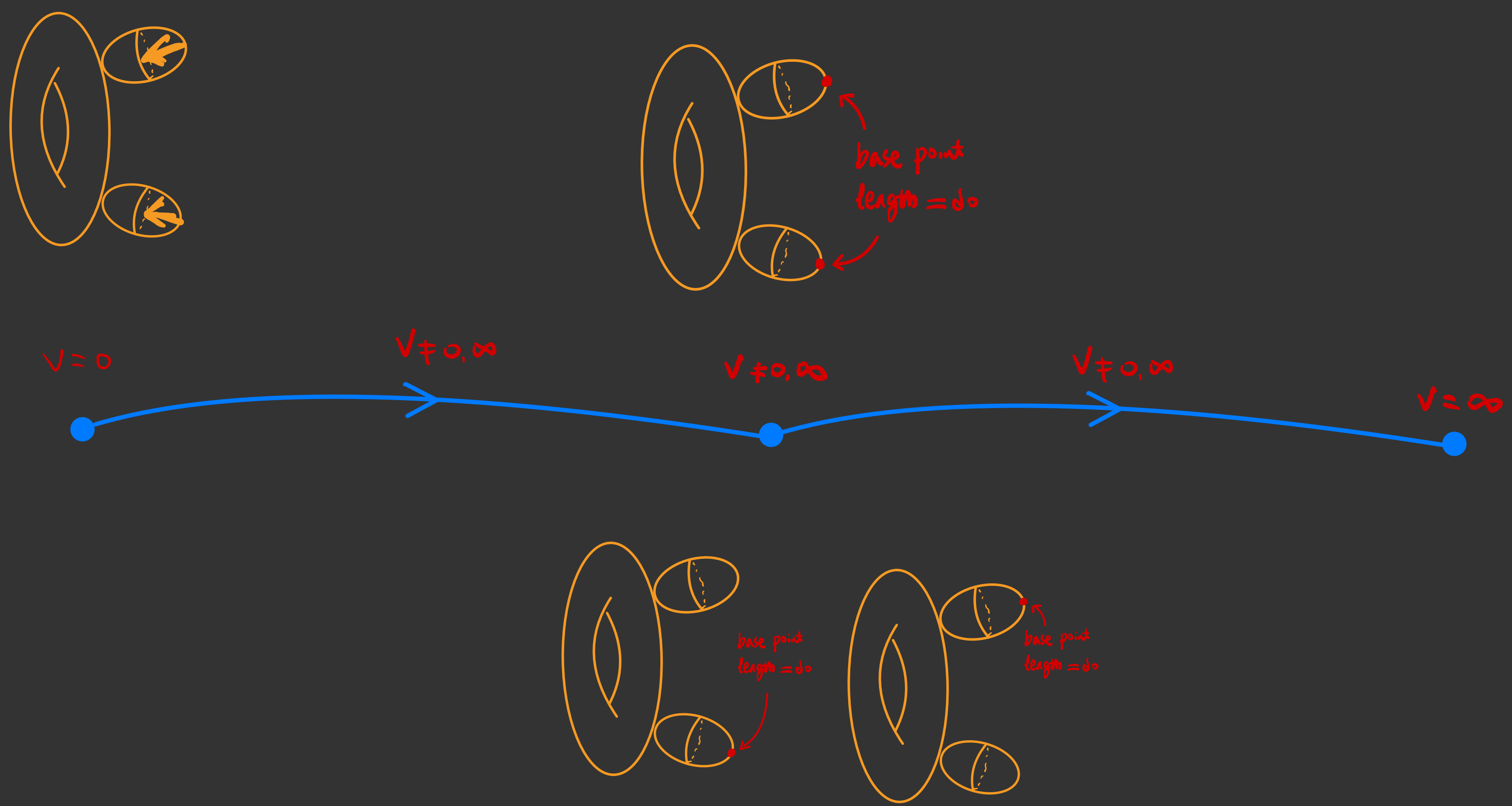
Θ_i is inf. smoothing of the i-th node (attached to a deg. d_0 tail)

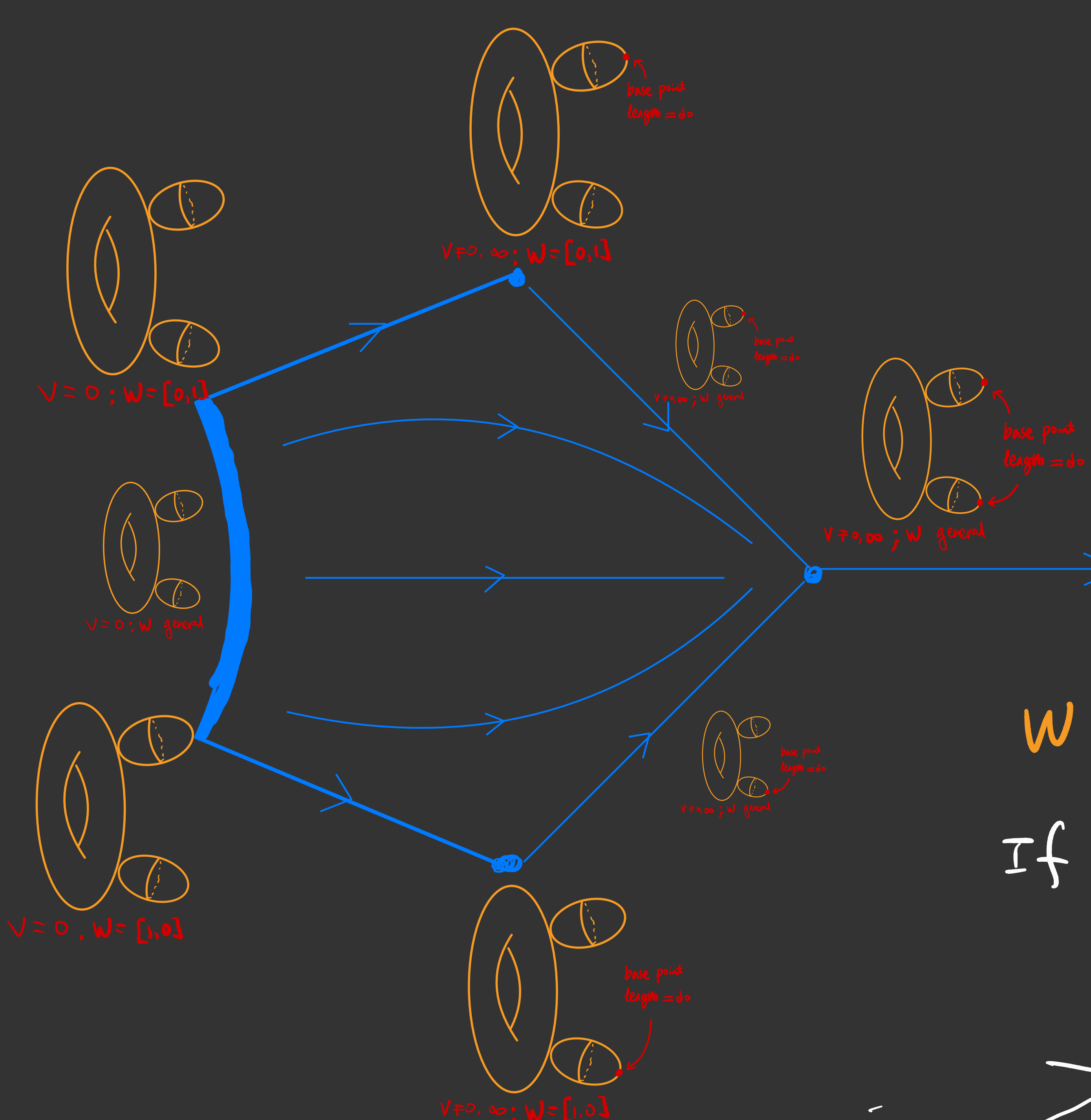
Rule 1: If $v=\infty$ then Σ_--stable. no tail of deg d_0.

Rule 2: If $v=0$, then Σ_+-stable, no. length d_0 bpts.

Σ_+ Σ_-

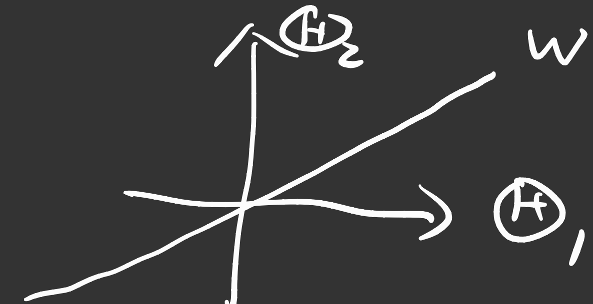



base point
length = d0
V = 0
V ≠ 0, ∞
V ≠ 0, ∞
V ≠ 0, ∞
V = ∞
base point
length = d0
base point
length = d0



$W \in \mathbb{P}(H_1 \oplus H_2)$

If $W \neq 0, \infty$



$W: H_1 \cong H_2$

= Entangled tails "